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Equations of motion in linearised gravity: I Uniform acceleration

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Abstract. We describe a straightforward approach to studying the motion of the sources of some Robinson–Trautman gravitational fields in linearised gravity. It involves expanding the Robinson–Trautman line-element about Minkowskian space–time in powers of a small parameter (the 'mass' of the source). We solve the linearised field equations *in vacuo* by first specifying the source world-line in the background Minkowskian space–time. Functions of integration are determined by the requirement that terms be excluded from the field (Riemann tensor) of the particle which are singular along null-rays emanating into the future from events on the source world-line in the background space–time. As an example we take the world-line to be the history of a uniformly accelerated particle. We show that our solution agrees with the exact solution of Levi-Civita to this problem, in the linear approximation.

1. Introduction

There are obvious similarities between the Robinson–Trautman (1962) solutions of the Einstein and Einstein–Maxwell vacuum field equations and the well-known Liénard–Wiechert solutions of Maxwell's vacuum field equations. Point-like sources of the former may therefore be expected to exhibit, under varying circumstances, geodesic motion, uniform acceleration, run-away motion, etc, in some well-defined technical sense.

The initial study of the possible motions of point-like sources of the Robinson–Trautman fields was carried out by Newman and Posadas (1969). Their approach, which involved expanding a key function in spherical harmonics, revealed no acceleration for an uncharged source in their first approximation. Nevertheless, there exists an exact solution of the vacuum Einstein field equations, discovered by Levi-Civita (1918) (for the recent history of this solution see Robinson and Robinson 1972), which is a member of the Robinson–Trautman family and has been interpreted by Kinnersley and Walker (1970) as representing the gravitational field of a uniformly accelerated point source. One would therefore expect to be able to identify such a solution in a linear or first approximation. This has been done by Robinson and Robinson (1972) using a method which emphasises the linearised field (Riemann tensor) of the particle rather than the linearised potentials (metric tensor components). Aside from the complexity of this approach, it is not clear how one might use it to study alternative types of motion in the linear approximation.

More recently, the concept of H space, introduced by Newman (1976) and Penrose (1976), has been used to study the problems we are concerned with in this paper (see Ludvigsen (1978) and references therein).

In the sequel we describe a method capable of dealing with a wide variety of types of motion, in the linear approximation. It entails firstly expanding the Robinson–Trautman form of the line-element about Minkowskian space–time in powers of the (small) mass parameter and arriving at the vacuum field equations to be satisfied by the perturbation of the metric from Minkowskian space–time. This is carried out in § 2. We also quote in § 2 a lemma on gauge transformations which we require for the integration of the linearised field equations in § 4. We review, in § 3, a geometrical construction of the relevant form for the Minkowskian background line-element mentioned in § 2. This form is not new. It appears to have first been obtained by Robinson (1963, private communication with the authors in Newman and Unti (1963)). The Minkowskian background contains a time-like world-line on which the perturbed metric is singular; it plays the role of the point-like source. In § 4 we specialise this world-line to have uniform acceleration, and this enables us to solve the differential equations outlined in § 2. We determine functions of integration by the requirement that we eliminate quantities which give rise to singularities in the field (Riemann tensor) of the particle along future null-rays emanating from the source world-line in the background Minkowskian space–time. We differ from previous work on this problem in that *we are specifying the motion and seeking the field*, whereas the contrary has been the case previously (cf Newman and Posadas 1969, Robinson and Robinson 1972, Ludvigsen 1978). These authors have incorporated additional assumptions, for one does not expect, in a linear field theory, to be able to extract the motion of the sources from the field equations alone. Finally, in § 5 we compare our results with the exact Levi-Civita (1918) solution, in the linear approximation.

2. Linearisation

We begin with the Robinson–Trautman form for the line-element,

$$ds^2 = 2r^2P^{-2}\, d\zeta\, d\bar{\zeta} - 2\, dr\, d\sigma - h\, d\sigma^2, \tag{2.1}$$

where ζ, $\bar{\zeta}$ are complex coordinates (the bar indicating complex conjugation), and r and σ are real coordinates. The function P is independent of r, while the function h depends on all four coordinates. Einstein's vacuum field equations imply (see Robinson and Trautman 1962)

$$h = K - 2Hr - 2m/r, \tag{2.2a}$$

$$K = \Delta \ln P \qquad (\Delta = 2P^2\, \partial^2/\partial\zeta\partial\bar{\zeta}), \tag{2.2b}$$

$$H = \partial(\ln P)/\partial\sigma, \tag{2.2c}$$

$$\tfrac{1}{4}\Delta K = -3Hm, \tag{2.2d}$$

where we have chosen m to be a constant without loss of generality. The constant m will be taken to be the mass of the linearised source (although it does not in general admit of such an interpretation in the exact theory). We shall assume that m is small (we write† $m = O_1$) and expand the function P as

$$P = \underset{0}{P} + \underset{1}{P} + O_2, \tag{2.3}$$

where $\underset{n}{P} = O_n$, $n = 0, 1, 2, \ldots$. We shall find it more convenient to write

$$P = \underset{0}{P}(1 + Q) + O_2, \tag{2.4}$$

where $Q = O_1$. Then

$$H = \underset{0}{H} + \underset{1}{H} + O_2, \tag{2.5a}$$

$$\underset{0}{H} = \partial(\ln \underset{0}{P})/\partial\sigma, \qquad \underset{1}{H} = \partial Q/\partial\sigma, \tag{2.5b}$$

$$K = \underset{0}{K} + \underset{1}{K} + O_2, \tag{2.5c}$$

$$\underset{0}{K} = \underset{0}{\Delta} \ln \underset{0}{P}, \qquad \underset{1}{K} = \underset{0}{\Delta} Q + 2\underset{0}{K} Q, \qquad \underset{0}{\Delta} = 2\underset{0}{P}^2\, \partial^2/\partial\zeta\partial\bar{\zeta}. \tag{2.5d}$$

If $m = 0$, then only the terms with a subscript zero remain, and we shall take them, namely $\underset{0}{P}$, $\underset{0}{H}$, $\underset{0}{K}$, to have the values necessary to make (2.1) the line-element of Minkowskian space–time (see § 3 for these values). Then, in particular, we will find that $\underset{0}{K} = 1$, and hence (2.2d) becomes

$$\underset{0}{\Delta}\underset{1}{K} = -12\underset{0}{H} m + O_2. \tag{2.6}$$

Having specified the source world-line $r = 0$ in the background Minkowskian space–time, we shall have at our disposal the functions $\underset{0}{P}$, $\underset{0}{H}$, $\underset{0}{K}$ from which we obtain $\underset{1}{K}$ using (2.6), and thence Q using (2.5d), and thence $\underset{1}{H}$ by (2.5b), the error in each case being O_2. Substituting these into (2.4), (2.5a) and (2.5c) we will then have a knowledge of the line-element (2.1) in the linear approximation, i.e. with an O_2 error. In carrying out this integration programme certain functions of integration will occur. We shall find the following lemma useful in dealing with them. We use the notation of Newman and Penrose (1962) for the tetrad components of the Riemann tensor.

Lemma. The linearised vacuum field equations (2.6), (2.5d) and (2.5b) and the tetrad components ψ_A ($A = 0, 1, 2, 3, 4$) of the corresponding linearised Riemann tensor are left invariant by the 'gauge transformation'.

$$Q \to Q + \tfrac{1}{2}A(\sigma) + B(\sigma, \zeta, \bar{\zeta}), \qquad \underset{1}{H} \to \underset{1}{H} + \tfrac{1}{2}\dot{A} + \dot{B}, \qquad \underset{1}{K} \to \underset{1}{K} + A \tag{2.7}$$

(the 'dot' indicating differentiation with respect to σ) *if*

$$\underset{0}{\Delta} B + 2B = 0, \tag{2.8a}$$

$$\partial(\underset{0}{P}^2\, \partial^2 B/\partial\zeta\partial\sigma + 2B\underset{0}{P}^2\, \partial\underset{0}{H}/\partial\zeta)/\partial\zeta = 0. \tag{2.8b}$$

This result is easily established by direct computation.

† We are using units for which $c = G = 1$. Hence strictly speaking we should have a parameter l (say), having the dimensions of length, so that $ml^{-1} = O_1$. In the sequel we shall have a source with uniform acceleration a and hence we would choose $l = a^{-1}$.

3. The background metric

Let X^i $(i=1,2,3,4)$ be rectangular Cartesian coordinates in the background Minkowskian space–time. Let $X^i = x^i(\sigma)$ be a time-like world-line C in this space–time with σ proper-time along it. Let $P(X)$ be a current event in the space–time, and let $Q(x)$ be the unique event of intersection of the past null-cone with vertex P and the world-line (see figure 1). Let $\lambda^i = \mathrm{d}x^i/\mathrm{d}\sigma$ be the tangent or four-velocity at Q, and let $\xi^i = X^i - x^i(\sigma)$; then $\lambda^i\lambda_i = -1$ and $\xi^i\xi_i = 0$, and if

$$r = -\lambda_i\xi^i \geqslant 0, \tag{3.1}$$

then r is the 'retarded distance', defined by Synge (1970), of P from the world-line C. It is zero if and only if P lies on C. Take $\mu^i = \mathrm{d}\lambda^i/\mathrm{d}\sigma$ to be the four-acceleration of C, and then $\lambda^i\mu_i = 0$. The construction above raises σ, λ^i, μ^i to the status of fields on Minkowskian space-time via the definitions $\sigma(X) \equiv \sigma(x)$, while λ^i, μ^i are postulated to be parallel transported along the null-line joining $Q(x)$ to $P(x)$. One can then calculate the following derivatives (see Synge 1970), the comma denoting partial differentiation with respect to X^i,

$$\begin{aligned} &\sigma_{,i} = -r^{-1}\xi_i, && \lambda_{i,j} = -r^{-1}\mu_i\xi_j, && \mu_{i,j} = -r^{-1}\nu_i\xi_j, \\ &(\nu^i = \mathrm{d}\mu^i/\mathrm{d}\sigma), && \xi_{i,j} = \eta_{ij} + r^{-1}\lambda_i\xi_j, && r_{,j} = -\lambda_j + B\xi_j, \end{aligned} \tag{3.2}$$

where η_{ij} is the metric tensor of Minkowskian space–time, and $B = r^{-1}(1+\mu^j\xi_j)$. From these equations one easily proves that

$$\mathrm{d}k^i/\mathrm{d}\sigma = (\mu^j k_j)k^i, \tag{3.3}$$

where $k^i = r^{-1}\xi^i$. Now k^i is tangent to the generators of the future-directed null-cones emanating from every event of C. We can choose r as parameter along these 'rays', i.e. write $k^i = \partial x^i/\partial r$, then it follows from (3.2) that r is an affine parameter along the rays, and the rays are, as one expects, geodesic, shear-free, twist-free and expanding, with expansion r^{-1}. Since μ^i, k^i are both parallel propagated along the null-rays, the scalar

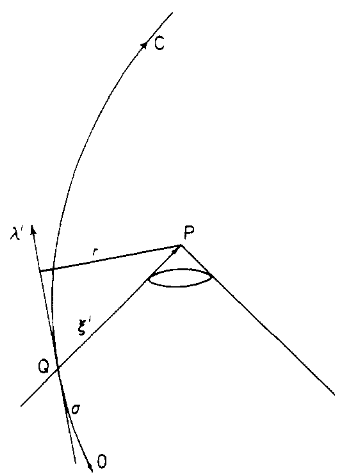

Figure 1. The retarded distance r of P from C. The event 0 is the origin of proper-time σ along C.

product $\mu^i k_i$ is independent of r. Hence if $\underset{0}{P} = \underset{0}{P}(\sigma, \zeta, \bar{\zeta})$, where ζ, $\bar{\zeta}$ are two complex coordinates, yet to be defined, then we may write

$$-\mu^i k_i = \partial(\ln \underset{0}{P})/\partial\sigma = \underset{0}{H}. \tag{3.4}$$

Then integrating (3.3) component by component we find that

$$k^i = \underset{0}{P}^{-1}\zeta^i(\zeta, \bar{\zeta}), \qquad \underset{0}{P} = -\lambda_i \zeta^i. \tag{3.5}$$

The second equation here follows from (3.1) by dividing by r. At an event $\sigma = \sigma_0$ (say) on the world-line C choose $\lambda^i = \delta^i_4$, i.e. λ^i is the time-axis at $\sigma = \sigma_0$; then, by (3.5), bearing in mind that ζ^i is a null-vector, we have

$$\zeta^4 = -\zeta_4 = \underset{0}{P}(\sigma_0, \zeta, \bar{\zeta}), \qquad \zeta_\alpha \zeta_\alpha = \underset{0}{P}^2(\sigma_0, \zeta, \bar{\zeta}), \tag{3.6}$$

where $\alpha = 1, 2, 3$. Hence $\underset{0}{P}^{-1}\zeta_\alpha$ is a unit three-vector in the three-flat orthogonal to λ^i at $\sigma = \sigma_0$. We may choose to parametrise it in the usual way with polar angles θ, ϕ with respect to the Cartesian basis of the three-flat, or, equivalently, we may use the complex coordinate $\zeta = \sqrt{2}\, e^{i\phi} \tan\frac{1}{2}\theta$ and its complex conjugate $\bar{\zeta}$. In this case we find, if in addition we choose (without loss of generality) $\underset{0}{P}(\sigma_0, \zeta, \bar{\zeta}) = 1 + \frac{1}{2}\zeta\bar{\zeta}$, that, for any value of σ,

$$k^i = \underset{0}{P}^{-1}[(1/\sqrt{2})(\zeta + \bar{\zeta})\delta^i_1 + (1/i\sqrt{2})(\zeta - \bar{\zeta})\delta^i_2 + (1 - \tfrac{1}{2}\zeta\bar{\zeta})\delta^i_3 + (1 + \tfrac{1}{2}\zeta\bar{\zeta})\delta^i_4], \tag{3.7a}$$

$$\underset{0}{P} = \underset{0}{P}(\sigma, \zeta, \bar{\zeta}) = \lambda^4(1 + \tfrac{1}{2}\zeta\bar{\zeta}) - \lambda^3(1 - \tfrac{1}{2}\zeta\bar{\zeta}) - (\zeta/\sqrt{2})(\lambda^1 - i\lambda^2) - (\bar{\zeta}/\sqrt{2})(\lambda^1 + i\lambda^2), \tag{3.7b}$$

using (3.5). This result appears to have been first found by Robinson (1963, private communication with the authors in Newman and Unti (1963)).

From the above equations we have

$$X^i = x^i(\sigma) + r\underset{0}{P}^{-1}\zeta^i. \tag{3.8}$$

If, in the manner of Newman and Unti (1963), we regard this as a coordinate transformation from the coordinates X^i to the coordinates ζ, $\bar{\zeta}$, r, σ, then a straightforward but tedious calculation reveals

$$\eta_{ij}\, dX^i\, dX^j = 2r^2\underset{0}{P}^{-2}\, d\zeta\, d\bar{\zeta} - 2\, dr\, d\sigma - (1 - 2\underset{0}{H}r)\, d\sigma^2, \tag{3.9}$$

where $\underset{0}{P} = \underset{0}{P}(\sigma, \zeta, \bar{\zeta})$ is given by (3.7b), and $\underset{0}{H}$ is given by (3.4). One can check, e.g. by calculating the Riemann tensor, that this is the line-element of Minkowskian space-time.

Comparison of (3.9) with (2.1) shows that the two line-elements coincide when $P = \underset{0}{P}$ and $h = 1 - 2\underset{0}{H}r$. One can easily show that, with $\underset{0}{P}$ given by (3.7b), $\underset{0}{\Delta} \ln \underset{0}{P} = 1$, and so $\underset{0}{K}$ given by (2.5d) is unity. We shall henceforth regard $\underset{0}{P}$ and $\underset{0}{H}$ given in this section and $\underset{0}{P}$ and $\underset{0}{H}$ of § 2 as synonymous.

4. Uniform acceleration

We here specialise the world-line C to have the equations

$$x^1 = x^2 = 0, \qquad x^3 = a^{-1}\cosh a\sigma, \qquad x^4 = a^{-1}\sinh a\sigma, \tag{4.1}$$

where a is a constant. In the background Minkowskian space–time this is the history of a particle having constant four-acceleration, i.e. $\mu_i\mu^i = a^2$, moving along the X^3 axis. By (3.7)

$$\underset{0}{P} = k_1(\tfrac{1}{2}\zeta\bar{\zeta} + k_2^2), \qquad \underset{0}{H} = a(\tfrac{1}{2}\zeta\bar{\zeta} - k_2^2)/(\tfrac{1}{2}\zeta\bar{\zeta} + k_2^2), \tag{4.2}$$

where

$$k_1 = \lambda^3 + \lambda^4 = e^{a\sigma}, \qquad k_2 = \lambda^3 - \lambda^4 = -e^{-a\sigma}, \tag{4.3}$$

and $k_1k_2 = -1$. Of the ten possible Killing vectors of the background space–time, one is clearly singled out here, namely, rotations about the X^3 axis or those transformations generated by the vector field.

$$i(\zeta\, \partial/\partial\zeta - \bar{\zeta}\, \partial/\partial\bar{\zeta}), \tag{4.4}$$

which is easily seen to be $\partial/\partial\phi$, where $\zeta = \sqrt{2}\, e^{i\phi} \tan\frac{1}{2}\theta$. It is reasonable to require that this symmetry be found in the space–time composed of the Minkowskian background and the small first-order perturbation. We guarantee this by requiring functions to depend on ζ and $\bar{\zeta}$ in the combination $\zeta\bar{\zeta}$ as in (4.2). It is then convenient to introduce, in place of $\zeta\bar{\zeta}$, the new variable

$$\underset{0}{\xi} = (\tfrac{1}{2}\zeta\bar{\zeta} - k_2^2)/(\tfrac{1}{2}\zeta\bar{\zeta} + k_2^2). \tag{4.5}$$

In terms of this we may rewrite (2.6) as

$$\partial[(1 - \underset{0}{\xi}^2)\partial \underset{1}{K}/\partial\underset{0}{\xi}]/\partial\underset{0}{\xi} = -12ma\underset{0}{\xi} + O_2. \tag{4.6}$$

This is easily integrated to give

$$\underset{1}{K} = 6ma\underset{0}{\xi} - A(\sigma) + \alpha(\sigma)\tfrac{1}{2}\ln[(1+\underset{0}{\xi})/(1-\underset{0}{\xi})] + O_2, \tag{4.7}$$

where A, α are functions of integration; they are here the coefficients of $l = 0$ Legendre functions of the first and second kind. It is convenient at this stage to examine the effect of these two terms on the tetrad components of the linearised Riemann tensor. We have, in general,

$$\begin{aligned} &\psi_0 = O_2, \qquad \psi_1 = O_2, \qquad \psi_2 = -m/r^3 + O_2, \\ &\psi_3 = \frac{1}{2}\frac{1}{r^2}\frac{\bar{\zeta}}{\underset{0}{P}}\frac{\partial \underset{1}{K}}{\partial\underset{0}{\xi}} + O_2, \\ &\psi_4 = \frac{1}{2r^2}\frac{\bar{\zeta}^2}{\underset{0}{P}^2}\frac{\partial^2 \underset{1}{K}}{\partial\underset{0}{\xi}^2} - \frac{1}{r}\frac{\bar{\zeta}^2}{\underset{0}{P}^2}\frac{\partial}{\partial\underset{0}{\xi}}\left(\frac{\partial \underset{1}{H}}{\partial\underset{0}{\xi}} + 2aQ\right) + O_2. \end{aligned} \tag{4.8}$$

On substituting (4.7) into ψ_3 we find

$$\psi_3 = \frac{3ma}{r^2}\frac{\bar{\zeta}}{\underset{0}{P}} + \frac{1}{2}\frac{\alpha}{r^2}\frac{\bar{\zeta}}{\underset{0}{P}}(1 - \underset{0}{\xi}^2)^{-1} + O_2. \tag{4.9}$$

This expression is not only singular at $r = 0$, as we would expect of the field of a point particle, but it is also singular when $\underset{0}{\xi} = \pm 1$. In the background Minkowskian space–

time $r = 0$ is, of course, the uniformly accelerated world-line C. However, by (3.4), (4.2) and (4.5), $\underset{0}{\xi} = \pm 1$ corresponds, in the background space–time, to a pair of generators k^i of the future null-cone at each event on C for which

$$\mu^i k_i = \pm a. \tag{4.10}$$

Thus, as well as being singular on C, ψ_3 is also singular on a pair of diametrically opposed future-directed null-rays at each event on C. Alternatively $\underset{0}{\xi} = -1$ corresponds to $\zeta\bar{\zeta} = 0$ or $\theta = 0$, while $\underset{0}{\xi} = +1$ corresponds to $\zeta\bar{\zeta} \to \infty$ or $\theta = \pi$, for each fixed value of σ (on each future null-cone on C). This is not the type of singularity structure one expects of the field of a simple pole particle, and we exclude it by taking $\alpha = 0$ in (4.7). As a result of this, the first term in the expression for ψ_4 (cf (4.8)) vanishes. The second of (2.5*d*) may be now written

$$\partial[(1-\underset{0}{\xi}^2)\partial Q/\partial\underset{0}{\xi}]/\partial\underset{0}{\xi} + 2Q = 6ma\underset{0}{\xi} - A(\sigma) + \mathrm{O}_2. \tag{4.11}$$

This is integrated in a standard way (see Bateman 1918, p 71) to give

$$Q = -ma\underset{0}{\xi}\ln(1-\underset{0}{\xi}^2) - \tfrac{1}{2}A(\sigma) - B(\sigma, \underset{0}{\xi}) + \mathrm{O}_2,$$

$$B = \beta(\sigma)\mathrm{P}_1(\underset{0}{\xi}) + \gamma(\sigma)\mathrm{Q}_1(\underset{0}{\xi}, \tag{4.12}$$

where $\mathrm{P}_1(\underset{0}{\xi})$, $\mathrm{Q}_1(\underset{0}{\xi})$ are the $l = 1$ Legendre functions of the first and second kind respectively, and $\beta(\sigma)$, $\gamma(\sigma)$ are functions of integration. Then $\underset{1}{H}$ is calculated by the second of (2.5*b*) using $\dot{\underset{0}{\xi}} = a(1-\underset{0}{\xi}^2)$. We then calculate ψ_4 given by (4.8) and find

$$\psi_4 = -\frac{6ma^2}{r}\frac{\bar{\zeta}^2}{\underset{0}{P}^2} + \frac{2}{r}\frac{\bar{\zeta}^2}{\underset{0}{P}^2}\dot{\gamma}(1-\underset{0}{\xi}^2)^{-2} + \mathrm{O}_2. \tag{4.13}$$

Both terms are singular on $r = 0$. The second term is also singular when $\underset{0}{\xi} = \pm 1$ unless $\dot{\gamma} = 0$. Since, as we have seen, $\underset{0}{\xi} = \pm 1$ is an unacceptable singularity in the field of a simple pole particle, we choose $\dot{\gamma} = 0$. Hence the tetrad components of the linearised Riemann tensor are finally given by

$$\psi_0 = \psi_1 = \mathrm{O}_2, \qquad \psi_2 = -m/r^3 + \mathrm{O}_2,$$

$$\psi_3 = \frac{3ma}{r^2}\frac{\bar{\zeta}}{\underset{0}{P}} + \mathrm{O}_2, \qquad \psi_4 = -\frac{6ma^2}{r}\frac{\bar{\zeta}^2}{\underset{0}{P}^2} + \mathrm{O}_2. \tag{4.14}$$

These are only singular on $r = 0$ and, in addition,

$$2\psi_3^2 - 3\psi_2\psi_4 = \mathrm{O}_3, \tag{4.15}$$

and so the linearised field of the particle is Petrov type D. We notice that the functions of integration remaining, $A(\sigma)$, $\beta(\sigma)$, γ, do not appear in (4.14). It is not surprising then that they can be removed by a guage transformation. The precise gauge transformation is given by (2.7). Clearly $B(\sigma, \underset{0}{\xi})$ given by (4.12) satisfies (2.8*a*). It also satisfies (2.8*b*) provided $\dot{\gamma} = 0$.

The 'directional' singularities which we have encountered in this section are also encountered, under a different guise, in the treatment of this problem by Robinson and Robinson (1972).

We note that, as $\sigma \to \pm\infty$, $\underset{0}{P}$ becomes infinite for either sign of a. Hence in the infinite past or infinite future the only non-vanishing (modulo and O_2 error) tetrad component of the Riemann tensor is $\psi_2 = -mr^{-3}$. Also the world-line C of the source becomes null in this limit, since the four-velocity components become finite. Hence the field of the particle in the infinite past or infinite future resembles a Robinson–Trautman dS space with $k=0$ (in their classification). In a certain sense such a solution represents the gravitational field of a particle travelling with the speed of light (see Hogan 1974). We also, of course, recover the linearised Schwarzschild solution from our results by putting $a=0$. This is a dS space, but with $k=1$.

5. Comparison with exact solution

We found in § 4 that the linearised field of a uniformly accelerated particle is described by the line-element (2.1) with

$$\begin{aligned} P &= -2k_2(1-\underset{0}{\xi})^{-1}[1 - ma\underset{0}{\xi}\ln(1-\underset{0}{\xi}^2)] + O_2, \\ H &= a\underset{0}{\xi} + ma^2[2\underset{0}{\xi}^2 - (1-\underset{0}{\xi}^2)\ln(1-\underset{0}{\xi}^2)] + O_2, \\ K &= 1 + 6ma\underset{0}{\xi} + O_2, \end{aligned} \tag{5.1}$$

with $\underset{0}{\xi}$ given by (4.5). We note that, although P here is singular on $\underset{0}{\xi} = \pm 1$, the field ψ_A is singular only on $r=0$. The exact Levi-Civita (1918) vacuum solution can be written in the form (see Kinnersley and Walker 1970)

$$ds^2 = r^2(G^{-1}\,d\xi^2 + G\,d\eta^2) - 2ar^2\,d\sigma\,d\xi - 2\,dr\,d\sigma - c\,d\sigma^2, \tag{5.2a}$$

where

$$G = 1 - \xi^2 - 2ma\xi^3 \tag{5.2b}$$

$$c = 1 + 6ma\xi - 2ar(\xi + 3ma\xi^2) - a^2r^2(1-\xi^2-2ma\xi^3) - 2mr^{-1}. \tag{5.2c}$$

The linearised form of this is

$$ds^2 = r^2(1-\xi^2)[1-2ma\xi^3/(1-\xi^2)][(b\,d\xi - a\,d\sigma)^2 + d\eta^2] - 2\,dr\,d\sigma - h\,d\sigma^2 + O_2, \tag{5.3a}$$

where

$$b = (1-\xi^2)^{-1}[1 + 2ma\xi^3/(1-\xi^2)], \tag{5.3b}$$

$$h = 1 + 6ma\xi - 2ar(\xi + 3ma\xi^2) - 2mr^{-1}. \tag{5.3c}$$

If one makes the transformation

$$\begin{aligned} \eta &= -\tfrac{1}{2}\mathrm{i}\ln(\zeta\bar{\zeta}^{-1}), \\ \xi &= \underset{0}{\xi} - ma\underset{0}{\xi}^2 - ma(1-\underset{0}{\xi}^2)\ln(1-\underset{0}{\xi}^2), \end{aligned} \tag{5.4}$$

with $\underset{0}{\xi}$ given by (4.5), one finds that (5.3a) takes the form (2.1) with P, H and K given by (5.1). Hence our approach leads to the linearised version of the exact Levi-Civita solution of Einstein's vacuum field equations for a uniformly accelerated mass.

The procedure for generalisation is now clear. The situation described in §§ 2 and 3 is quite general. One begins solving the linearised field equations (2.6), (2.5*d*) and (2.5*b*) once the world-line C in the background Minkowskian space–time has been specified. One may not, however, always have sufficient functions of integration at one's disposal to remove all of the 'directional' singularities that may crop up. Robinson and Robinson (1972) have given conditions which have the effect that only the case of uniform acceleration is free of directional singularities. A 'run-away' uncharged source (see Hogan and Imaeda 1979) is an example in which one can remove one or other but not both of the directional singularities on each future null-cone on C.

The question of what provides the acceleration of the particle is a difficult one. Kinnersley and Walker (1970) have shown that in the exact solution the two-surfaces $r = \text{constant}$, $\sigma = \text{constant}$, possess conical singularities at the north or south poles. They conjecture that these are related to the fact that one does not take account of an external field to drive the particle. Recently Ernst (1978) has suggested how one might append an external gravitational field and remove the conical singularity. The approximate solution we have described in this paper also has such a conical singularity. Problems concerning the physical interpretation of this are discussed in detail by us in a forthcoming paper.

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